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Abstract
In this study, we show that the energy eigenvalues and the eigenfunctions of the
Schrödinger equation for the power law and the logarithmic potentials can be
easily obtained by using a variation technique for special type wavefunctions.
The results are in very good agreement with exact numerical results.

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1. Introduction

A large number of important physical problems require solving the Schrödinger equation for
a spherical symmetric potential to determine the energy eigenvalues and the eigenfunctions.
It is known that for very limited potentials, the Schrödinger equation is exactly solvable. In
general, one has to resort to numerical techniques or approximation schemes. Most popular
approximation methods such as $1/N$ expansion, WKB method, perturbation theory are widely
used for this purpose. But some of these methods have drawbacks in application. Although
some methods give simple relations for the eigenvalues, they give very complicated relations
for the eigenfunctions. The aim of this work is to give a simple way for finding both the
eigenvalues and the eigenfunctions of Schrödinger and Schrödinger-like equations for the
power law and the logarithmic potentials, which are very important in particle physics [1–10].
This paper is organized as follows. In the first section, the eigenvalues are obtained for
a $\mathrm{sgn}(v)Ar^v(v > -2)$ type potential. Using the similarity of the exact solution of the
Schrödinger equation for $v = -1$ and $v = 2$, we have proposed a trial wavefunction with two
parameters, and then, in the framework of the standard variation technique, we have obtained
eigenvalues of the power law and the logarithmic potentials for the Schrödinger equation.
Because of the structure of our trial wavefunction and general characteristics of variational
calculations, we obtain an upper bound to the eigenvalues. We note that a trial wavefunction
with two parameters gives better accurate results for the eigenvalues than a wavefunction with
one parameter. In section 2, the same technique is applied to the logarithmic potential. In the
last section, we give some concluding remarks.

2. The solution of the Schrödinger equation for the power-law potentials

The radial part of Schrödinger equation for the $\text{sgn}(v)Ar^v (v > -2)$ type potential is written as

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \text{sgn}(v)Ar^v + \frac{\hbar^2 l(l+1)}{2mr^2} \right] g(r) = Eg(r). \tag{1}$$

Using substitution $r = \left(\frac{\hbar^2}{2mA}\right)^{\frac{1}{v+2}} \rho$, equation (1) can be written in the following form,

$$\left[-\frac{d^2}{d\rho^2} + \text{sgn}(v)\rho^v + \frac{l(l+1)}{\rho^2} \right] g(\rho) = \varepsilon g(\rho) \tag{2}$$

where $\varepsilon = E\left(\frac{1}{A}\right)^{\frac{2}{v+2}}\left(\frac{\hbar^2}{2m}\right)^{-\frac{v}{v+2}}$. It is well known that equation (2) has exact analytical solutions for $v = -1$ and $v = 2$ and the forms of the solutions are:

- for $v = -1$ (Coulomb potential) with $x = \frac{1}{2(n+l+1)}$, $(n = 0, 1, 2, \dots$ and $l = 0, 1, 2, \dots)$

$$g(\rho) = \rho^{l+1} \exp(-x\rho)L_n^{2l+1}(2x\rho) \tag{3}$$

- for $v = 2$ (harmonic potential) with $x = \frac{1}{\sqrt{2}}$

$$g(\rho) = \rho^{l+1} \exp(-(x\rho)^2)L_n^{\frac{2l+1}{2}}(2(x\rho)^2) \tag{4}$$

where L are Laguerre polynomials. From the similarity of these solutions, the following solution can be proposed for arbitrary v

$$g(\rho) \approx \rho^{l+1} \exp(-(x\rho)^d)L_n^{\frac{2l+1}{d}}(2(x\rho)^d) \tag{5}$$

where x and d are variation parameters and they can be obtained by minimizing ϵ in equation (2) with respect to these parameters. So

$$\frac{\partial \varepsilon}{\partial x} = 0 \tag{6}$$

$$\frac{\partial \varepsilon}{\partial d} = 0. \tag{7}$$

Using equations (2) and (5), for ϵ, we get

$$\varepsilon_{nl}(x, d) = cx^2 + \frac{b}{x^v} \tag{8}$$

and from equation (6), we have $x = \left(\frac{bv}{2c}\right)^{\frac{1}{v+2}}$, thus, we have found ε_{nl} in the following form:

$$\varepsilon_{nl}(d) = (v+2)\left(\frac{c}{v}\right)^{\frac{v}{v+2}}\left(\frac{b}{2}\right)^{\frac{2}{v+2}}. \tag{9}$$

Thus, eigenvalues of equation(1) can be written as follows

$$E_{nl} = \left(\frac{\hbar^2}{2m}\right)^{\frac{v}{v+2}}(A)^{\frac{2}{v+2}}\varepsilon_{nl} \tag{10}$$

where

$$c = 2^{\frac{2-2d}{d}}\frac{\sum_{k=0}^{n}\sum_{m=0}^{n} a_k a_m s\Gamma\left(k+m+\frac{2l+1}{d}\right)}{\sum_{k=0}^{n}\sum_{m=0}^{n} a_k a_m \Gamma\left(k+m+\frac{2l+3}{d}\right)} \tag{11}$$

Figure 1. The dependence of d on v.

$$b = \text{sgn}(v)2^{\frac{-v}{d}} \frac{\sum_{k=0}^{n} \sum_{m=0}^{n} a_k a_m \Gamma\left(k + m + \frac{2l+v+3}{d}\right)}{\sum_{k=0}^{n} \sum_{m=0}^{n} a_k a_m \Gamma\left(k + m + \frac{2l+3}{d}\right)} \tag{12}$$
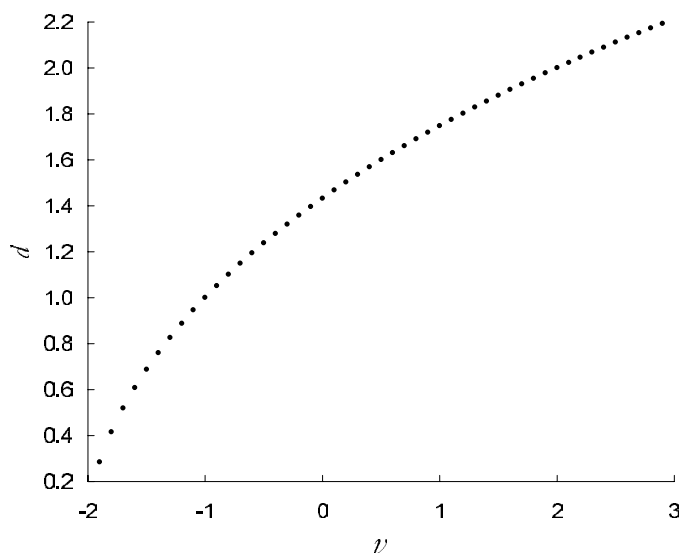
$$s = (2l + 1)(2l + d + 1) + (k + m - (k - m)^2)d^2 \tag{13}$$

where a_j are the coefficients of the generalized Laguerre polynomials. The parameter d can be obtained from equation (7), but unfortunately this equation cannot be solved analytically. However, for the $n = 0, l = 0$ case, the value of d can be found easily by minimizing ε. In figure 1, we present the dependence d on v. The behaviour of the curve is similar to the $\sqrt{v + 2}$ function. But, this function does not fit exactly to the curve. Therefore, a correction factor is necessary. According to our assumption, the correction factor must be equal to 1 at $v = -1$ and $v = 2$. Thus, we choose the correction factor in the following form:

$$w = (1 + tp)^h \tag{14}$$

where $p = \frac{(v+1)(2-v)}{a_1 v^2 + a_2 v + a_3}$. So, we write

$$d = \sqrt{v + 2}\, w. \tag{15}$$

Fitting this equation to the curve in figure 1, t, a_1, a_2, a_3 and h constants are obtained as 0.2075, 0.1381, 1.05, 2.484 and 0.081 04, respectively. So, when any values of v are given, d values corresponding to v can be easily calculated by using equation (15). It should be noted that the eigenvalues obtained by equation (9) are always upper bounds to the exact eigenvalues due to the structure of used trial wavefunction and the general characteristic of variation calculations. In other words, our approximation is always on the high side becoming lower. In tables 1 and 2, we present the results of our calculations for the eigenvalues at different v. In these tables, for comparison, we also present the existing numerical solutions in the literature.

When we study linear potentials, equation (2) has the following form for S states:

$$\left[-\frac{d^2}{dz^2} + z \right] g(z) = 0 \tag{16}$$

Table 1. Comparison of this study results for the ground state ($n = 0, l = 0$) of various power-law potentials ($\frac{\hbar^2}{2m} = 1$).

$V(r)$	This work	Numerical [12]	$V(r)$	This work	Numerical [12]
$-r^{-1.5}$	$-0.297\,03$	$-0.298\,59^a$	r^2	3	3
$-r^{-1.25}$	$-0.220\,27$	$-0.220\,37^a$	r^3	3.451\,10	3.450\,56
$-r^{-1}$	-0.25	-0.25	r^4	3.802\,41	3.799\,67
r^0	1	1	r^5	4.096\,26	4.338\,01
$r^{0.15}$	1.327\,98	1.327\,95	r^6	4.352\,43	4.546\,90
$r^{0.5}$	1.833\,52	1.833\,39	r^7	4.581\,58	4.717\,72
$r^{0.75}$	2.108\,29	2.108\,14	r^8	4.790\,13	4.922\,20
$r^{1.5}$	2.708\,16	2.708\,09	r^{10}	5.160\,92	–

^{a} Result computed by Q Katatbeh using Hall's valuable programs for computing exact eigenvalues.

Table 2. Eigenvalues of $-2^{1.7}r^{-0.2}$ and $-2^{0.8}r^{-0.8}$ for different nl ($\frac{\hbar^2}{m} = 1$).

n	l	This work	Numerical [12–16]	n	l	This work	Numerical [12–16]
0	0	-2.6859	-2.686	0	0	-1.2186	-1.218
1	0	-2.2530	-2.253	1	0	-0.4622	-0.462
2	0	-2.0440	-2.044	2	0	-0.2648	-0.265
0	1	-2.3449	-2.345	0	1	-0.5004	-0.500
1	1	-2.1006	-2.101	1	1	-0.2806	-0.281
2	1	-1.9504	-1.951	2	1	-0.1873	-0.187
0	2	-2.1562	-2.156	0	2	-0.2947	-0.295
1	2	-1.9900	-1.990	1	2	-0.1949	-0.195
2	2	-1.8749	-1.875	2	2	-0.1420	-0.142
0	3	-2.0291	-2.029	0	3	-0.2019	-0.202
1	3	-1.9049	-1.905	1	3	-0.1463	-0.146
2	3	-1.8124	–	2	3	-0.1128	–

Table 3. Eigenvalues of linear potential for different n and $l = 0$ ($\frac{\hbar^2}{2m} = 1$).

n	l	This work	Numerical [11, 17]
0	0	2.338\,25	2.338\,10
1	0	4.089\,18	4.087\,95
2	0	5.521\,32	5.520\,56
3	0	6.786\,14	6.786\,71
4	0	7.941\,89	7.944\,13
5	0	9.018\,59	9.022\,65

where $z = \rho - \epsilon$. The equation given above is known as the Airy equation and has an exact solution in terms of Airy functions given in the following form [11]

$$g_n^{\text{exact}}(\rho) = \text{Ai}(\rho - \varepsilon_n) \tag{17}$$

where ε_n are the zeros of the Airy function and are as given in table 3. For S states, our prediction for eigenfunctions has the following form from equation (5):

$$g_n^{\text{our}}(\rho) \approx \rho^l \exp(-(x\rho)^d) L_n^{\frac{1}{d}}(2(x\rho)^d). \tag{18}$$

In figures 2 and 3, our wavefunction and the exact wavefunction are presented together for $n = 0, l = 0$ and $n = 4, l = 0$ states, respectively. We see that the agreement between the two solutions is excellent. In addition to the wavefunctions, in table 3, eigenvalues of the
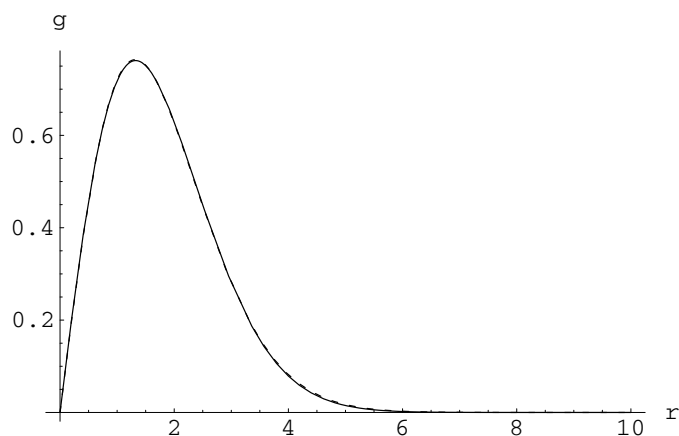
Figure 2. Comparison of our wavefunction and the Airy function for linear potential ($n = 0, l = 0$). The dotted line represents the Airy function.
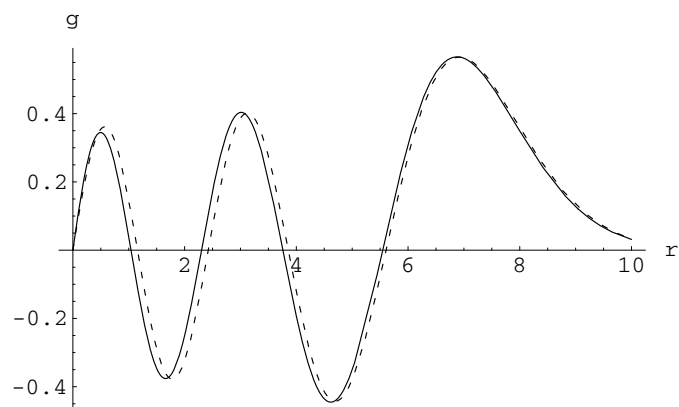

Figure 3. Comparison of our wavefunction and the Airy function for linear potential ($n = 4, l = 0$). The dotted line represents the Airy function.

linear potential for S states ($n = 0, 1, 2, \ldots$ and $l = 0$) are given and compared with the exact results which are well-known zeros of the Airy function. Similarly, in table 4, eigenvalues of $r^{0.5}$ are calculated and compared with the exact numerical results. The calculated results are in good agreement with the exact numerical results and better than those given in [18].

3. The logarithmic potential case

Let us consider $V(r) = \log(r)$ potential which is very important in particle physics. The radial part of the Schrödinger equation for the logarithmic potential is written as

$$\left[-\frac{d^2}{dr^2} + \log(r) + \frac{l(l+1)}{r^2} \right] g(r) = E g(r). \tag{19}$$

The function $\log(r)$ at $\nu \cong 0$ can be written in the following form:

$$\log(r) \cong \frac{1}{\nu}[r^\nu - 1]. \tag{20}$$

Table 4. Eigenvalues of $r^{0.5}$ for different nl together with exact values and results of other researchers, with percentage errors ($\frac{\hbar^2}{2m} = 1$).

n	l	This work	Numerical [18]	[18]	%
0	0	1.833 52	1.833 39	1.833 75	0.007
1	0	2.551 52	2.550 65	2.551 42	0.03
2	0	3.051 77	3.051 18	3.052 24	0.019
3	0	3.451 97	3.452 13	3.453 41	0.005
4	0	3.792 33	3.793 36	3.794 82	0.027
0	1	2.300 56	2.300 50	2.300 73	0.003
1	1	2.854 73	2.854 34	2.854 86	0.014
2	1	3.286 66	3.285 83	3.286 59	0.025
3	1	3.648 38	3.647 39	3.648 35	0.027
4	1	3.963 61	3.962 68	3.963 82	0.023
0	2	2.657 60	2.657 56	2.657 75	0.002
1	2	3.120 48	3.120 33	3.120 77	0.005
2	2	3.502 96	3.502 45	3.503 09	0.015
3	2	3.833 38	3.832 54	3.833 36	0.022
4	2	4.126 86	4.125 81	4.126 78	0.025
0	3	2.954 48	2.954 45	2.954 61	0.001
1	3	3.357 64	3.357 59	3.357 98	0.001
2	3	3.702 99	3.702 70	3.703 27	0.008
3	3	4.007 96	4.007 37	4.008 10	0.015
4	3	4.282 82	4.281 96	4.282 83	0.020
0	4	3.212 36	3.212 33	3.212 47	0.001
1	4	3.572 75	3.572 75	3.573 10	0.000
2	4	3.889 13	3.888 98	3.889 50	0.004
3	4	4.173 08	4.172 68	4.173 35	0.010
4	4	4.431 96	4.461 31	4.431 64	0.015

Table 5. Eigenvalues of the logarithmic potential for different nl ($\frac{\hbar^2}{2m} = 1$).

n	l	This work	Numerical [12–16]	n	l	This work	Numerical [12–16]
0	0	1.0445	1.0443	3	0	2.5957	2.5957
0	1	1.6412	1.6430	3	1	2.7465	2.7440
0	2	2.0134	2.0150	3	2	2.8801	2.8800
0	3	2.2842	2.2860	3	3	2.9996	2.9990
1	0	1.8485	1.8474	3	4	3.1071	3.1070
1	1	2.1513	2.1510	4	0	2.8293	2.8299
1	2	2.3875	2.3880	4	1	2.9498	2.9480
1	3	2.5798	2.5810	4	2	3.0592	3.0600
2	0	2.2903	2.2897	4	3	3.1592	3.1590
2	1	2.4917	2.4910	4	4	3.2512	3.2510
2	2	2.6629	2.6630	6	0	3.1770	3.1791
2	3	2.8106	–	10	0	3.6411	3.6427

Substituting equation (20) into equation (19) and then applying the method presented in the previous section to equation (19) for d we found the value 1.432 03 at $\nu = 0$ (see equation (15)). Thus, the eigenvalues of equation (19) are written as

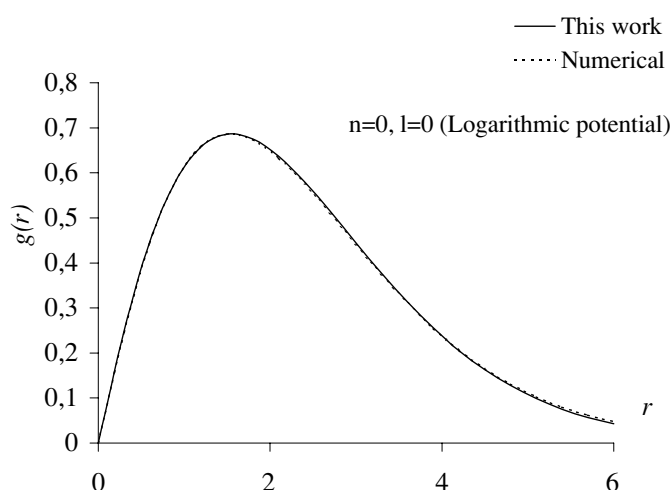$$E_{nl} = \frac{\varepsilon_{nl}}{\nu^{\frac{2}{\nu+2}}} - \frac{1}{\nu} \tag{21}$$

Figure 4. Comparison of our wavefunction and the corresponding numerical wavefunction for the logarithmic potential ($n = 0, l = 0$).
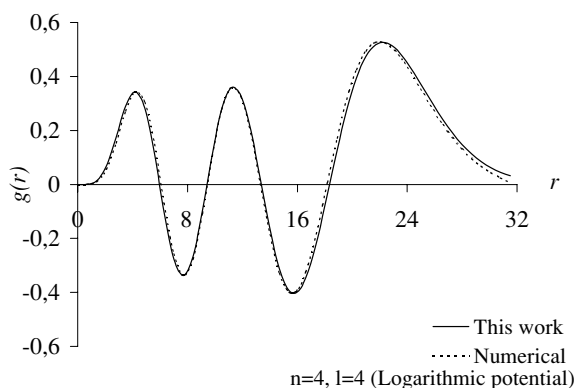

Figure 5. Comparison of our wavefunction and the corresponding numerical wavefunction for the logarithmic potential ($n = 4, l = 4$).

where ε_{nl} is given as in equation (9). The calculated results for E_{nl} are presented in table 5. In this table we also give the results of numerical solutions. The eigenfunctions of logarithmic potential can be written from equation (5) as

$$g(\rho) \approx \rho^{l+1} \exp(-(x\rho)^d) L_n^{\frac{2l+1}{d}} (2(x\rho)^d) \tag{22}$$

where $\rho = \dfrac{r}{\nu^{\frac{1}{\nu+2}}}$. It is obvious that the eigenvalues obtained for the logarithmic potential are in good agreement with the results obtained from the numerical solution. Also, in order to show the validity of the wavefunctions given in equation (22) obtained for the logarithmic potential, first of all, some energy levels are chosen. Their wavefunctions are obtained by Numerov's method. Finally, these results are given in figures 4 and 5 together with predictions.

4. Conclusion

In this paper, we have calculated eigenvalues and eigenfunctions of power law and logarithmic potentials by using variational techniques for special type wavefunctions. Our results are in good agreement with the existing exact numerical ones. However, for the higher values of ν, there are some differences between the two approaches. However, these differences are no larger than those coming from $1/N$ expansion. Moreover, we obtained that the present method predicts not only the eigenvalues but also as the eigenfunctions of given potentials. The correspondence of wavefunctions for lower energy levels is in very good agreement with the numerical results as expected except when n and l are large. The method used in this study presents an easy way of calculating both eigenfunctions and eigenvalues of the power law and the logarithmic potentials. In spite of its simple structure, the method is very practical and the results of the method are in good agreement with exact results. When we consider the tables, especially table 1, it is seen that some results of [12] are higher than ours. Because our results are upper bounds to the eigenvalues, it can be said that the results of [12] can be considered upper estimates.

Acknowledgments

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